7.1.12

(a) Let $\Psi(\theta) = \theta^2$. Then, $\Psi'(\theta) = 2\theta$ and $\Psi^{-1}(\psi) = \psi^{1/2}$. By Theorem 2.6.2, $\pi_\Psi(\psi) = \pi(\Psi^{-1}(\psi))/\Psi'^{-1}(\psi) = 0.5\psi^{-1/2}$. Thus, π_Ψ is not uniform on $[0,1]$.

(b) As we can see in part (a), complete ignorance is not achieved for an arbitrary function of a parameter, at least when we demand that a distribution be uniform to reflect ignorance. Notice, however, that Ψ is 1-1 and the change from a uniform distribution for θ to a nonuniform distribution for ψ is caused by the change of variable factor $\psi^{-1/2}$ which reflects how the transformation Ψ is changing lengths (Ψ shortens lengths more severely for intervals near 0.)

7.2.1 Recall that for the model discussed in Example 7.1.1, the posterior distribution of θ was Beta$(n\bar{x}+\alpha, n(1-\bar{x})+\beta)$. The posterior density is then given by

$$\pi_{\theta|x_1,\ldots,x_n} = \frac{\Gamma(\alpha+\beta+n)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)}\theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}$$

The posterior mean is given by

$$\begin{aligned} &E(\theta^m \mid x_1,\ldots,x_n) \\ &= \int_0^1 \frac{\Gamma(\alpha+\beta+n)}{\Gamma(n\bar{x}+\alpha)\Gamma(n(1-\bar{x})+\beta)}\theta^{n\bar{x}+\alpha+m-1}(1-\theta)^{n(1-\bar{x})+\beta-1}\,d\theta \\ &= \frac{\Gamma(\alpha+\beta+n)\Gamma(n\bar{x}+\alpha+m)}{\Gamma(n\bar{x}+\alpha)\Gamma(\alpha+\beta+n+m)}. \end{aligned}$$

7.2.2 Recall that for the model discussed in Example 7.1.2 the posterior distribution of μ is

$$N\left(\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right), \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right)$$

By exercise 2.6.3, the posterior distribution of the third quartile $\Psi = \mu+\sigma_0 z_{0.75}$ is

$$N\left(\left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right)+\sigma_0 z_{0.75}, \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\right)$$

Since the normal distribution is symmetric about its mode and the mean exists, the posterior mode and mean agree and given by

$$\hat{\psi} = \left(\frac{1}{\tau_0^2}+\frac{n}{\sigma_0^2}\right)^{-1}\left(\frac{\mu_0}{\tau_0^2}+\frac{n}{\sigma_0^2}\bar{x}\right)+\sigma_0 z_{0.75}.$$

7.2.4 Recall that the posterior distribution of σ^2 in Example 7.2.1 is inverse Gamma$(\alpha_0 + n/2, \beta_x)$, where β_x is given by (7.1.8). The posterior mean is then given by

$$E\left(\sigma^2 \,|\, x_1, ..., x_n\right) = \int_0^\infty \frac{1}{y} \frac{\beta_x^{\alpha_0+n/2}}{\Gamma\left(\alpha_0+n/2\right)} y^{\alpha_0+n/2-1} e^{-\beta_x y}\, dy$$
$$= \frac{\beta_x^{\alpha_0+n/2}}{\Gamma\left(\alpha_0+n/2\right)} \int_0^\infty y^{\alpha_0+n/2-2} e^{-\beta_x y}\, dy$$
$$= \frac{\beta_x^{\alpha_0+n/2}}{\Gamma\left(\alpha_0+n/2\right)} \frac{\Gamma\left(\alpha_0+n/2-1\right)}{\beta_x^{\alpha_0+n/2-1}} \int_0^\infty \frac{1}{\Gamma\left(\alpha_0+n/2-1\right)} y^{\alpha_0+n/2-2} e^{-y}\, dy$$
$$= \frac{\beta_x}{\alpha_0+n/2-1}.$$

By Theorem 2.6.2 the posterior density of σ^2 is given by $\pi\left(\sigma^2 \,|\, x_1, .., x_n\right) = \left(\Gamma\left(\alpha_0+n/2\right)\right)^{-1} \left(\beta_x\right)^{\alpha_0+n/2} \left(\sigma^2\right)^{-(\alpha_0+n/2+1)} \exp\left(-\beta_x/\sigma^2\right)$. Then to find the posterior mode we need only maximize $\ln\left(y^{-(\alpha_0+n/2+1)} \exp\left(-\beta_x/y\right)\right) = -\left(\alpha_0+n/2+1\right)\ln y - \beta_x/y$. This has first derivative given by

$\beta_x/\left(\alpha_0+n/2+1\right)$. The second derivative at this value is $\left(\alpha_0+n/2+1\right)^2/\beta_x^2 - 2\left(\alpha_0+n/2+1\right)^3/\beta_x^2 = \left(\alpha_0+n/2+1\right)^2\left(-1-2\alpha_0-n\right)/\beta_x^2 < 0$, so this is the unique mode.

7.2.6 Recall that the posterior distribution of θ in Example 7.2.2 is Beta$(n\bar{x}+\alpha, n\left(1-\bar{x}\right)+\beta)$. To find the posterior variance we need only to find the second moment as follows.

$$E\left(\theta^2 \,|\, x_1, ..., x_n\right)$$
$$= \int_0^1 \theta^2 \frac{\Gamma\left(n+\alpha+\beta\right)}{\Gamma\left(n\bar{x}+\alpha\right)\Gamma\left(n\left(1-\bar{x}\right)+\beta\right)} \theta^{n\bar{x}+\alpha-1}\left(1-\theta\right)^{n(1-\bar{x})+\beta-1}\, d\theta$$
$$= \frac{\Gamma\left(n+\alpha+\beta\right)}{\Gamma\left(n\bar{x}+\alpha\right)\Gamma\left(n\left(1-\bar{x}\right)+\beta\right)} \int_0^1 \theta^{n\bar{x}+\alpha+1}\left(1-\theta\right)^{n(1-\bar{x})+\beta-1}\, d\theta$$
$$= \frac{\Gamma\left(n+\alpha+\beta\right)}{\Gamma\left(n\bar{x}+\alpha\right)\Gamma\left(n\left(1-\bar{x}\right)+\beta\right)} \frac{\Gamma\left(n\bar{x}+\alpha+2\right)\Gamma\left(n\left(1-\bar{x}\right)+\beta\right)}{\Gamma\left(n+\alpha+\beta+2\right)}$$
$$= \frac{\left(n\bar{x}+\alpha+1\right)\left(n\bar{x}+\alpha\right)}{\left(n+\alpha+\beta+1\right)\left(n+\alpha+\beta\right)}$$

The posterior variance is then given by

$$\begin{aligned}
\operatorname{Var}(\theta \mid x_1, ..., x_n) &= E(\theta^2 \mid x_1, ..., x_n) - (E(\theta \mid x_1, ..., x_n))^2 \\
&= \frac{(n\bar{x} + \alpha + 1)(n\bar{x} + \alpha)}{(n + \alpha + \beta + 1)(n + \alpha + \beta)} - \left(\frac{n\bar{x} + \alpha}{n + \alpha + \beta}\right)^2 \\
&= \frac{(n\bar{x} + \alpha)(n(1 - \bar{x}) + \beta)}{(n + \alpha + \beta + 1)(n + \alpha + \beta)^2}.
\end{aligned}$$

Now $0 \le \bar{x} \le 1$, so

$$\begin{aligned}
\operatorname{Var}(\theta \mid x_1, ..., x_n) &= \frac{(n\bar{x} + \alpha)(n(1 - \bar{x}) + \beta)}{(n + \alpha + \beta + 1)(n + \alpha + \beta)^2} \\
&\le \frac{(1 + \alpha/n)(1 + \beta/n)}{n(1 + \alpha/n + \beta/n + 1/n)(1 + \alpha/n + \beta/n)^2} \to 0
\end{aligned}$$

as $n \to \infty$.

7.2.10 The likelihood function is given by $L(\lambda \mid x_1, ...x_n) = \lambda^n e^{-n\bar{x}\lambda}$. The prior distribution has density given by $\beta_0^{\alpha_0}\lambda^{\alpha_0 - 1}e^{-\beta_0\lambda}/\Gamma(\alpha_0)$. The posterior density of λ is then given by $\pi(\lambda \mid x_1, ...x_n) \propto \lambda^{n+\alpha_0-1}e^{-\lambda(n\bar{x}+\beta_0)}$, and we recognize this as being the density of a Gamma$(n + \alpha_0, n\bar{x} + \beta_0)$ distribution. The posterior mean and variance of λ are then given by $E(\lambda \mid x_1, ...x_n) = (n + \alpha_0)/(n\bar{x} + \beta_0)$, $Var(\lambda \mid x_1, ...x_n) = (n + \alpha_0)/(n\bar{x} + \beta_0)^2$.

To find the posterior mode we need to maximize $\ln\left(\lambda^{n+\alpha_0-1}e^{-\lambda(n\bar{x}+\beta_0)}\right) = (\alpha_0 + n - 1)\ln\lambda - \lambda(n\bar{x} + \beta_0)$. This has first derivative given by $(\alpha_0 + n - 1)/\lambda - (n\bar{x} + \beta_0)$ and second derivative $-(\alpha_0 + n - 1)/\lambda^2$. Setting the first derivative equal to 0 and solving gives the solution $\hat{\lambda} = (\alpha_0 + n - 1)/(n\bar{x} + \beta_0)$. The second derivative at this value is $-(n\bar{x} + \beta_0)^2/(\alpha_0 + n - 1)$, which is clearly negative, so $\hat{\lambda}$ is the unique posterior mode.